A countable Model of Set Theory

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1 Summary

This paper describes how a countable domain can span an uncountable universe of Neumann-Bernays-Gödel set theory (NBG).
We start by choosing a countable domain, map the '$\in$' and 'is a set' relations to it and assume the set axioms for granted. We then proof, that the NBG-axioms are satisfied within this countable domain. The items of the universe are formulars with one free parameter. The formulars have a dual role. On one hand they are simply labels for classes, on the other hand their logic provides the mapping rules to establish the element relationship.

2 Introduction

The theory of the is article is based on $\mathcal{L}I$ propositional logic, the Peano axioms and the core set axioms. The latter will be introduced during the construction as mapping rules. In this writing we are based on Neumann-Bernay-Gödel (NBG) set theory as defined in [Gödel]. For the basics in mathematical logic we follow [Ebbinghaus]. The language $\mathcal{L}_{NBG}$ of it consists of the symbols $\wedge, \vee, \rightarrow, \leftrightarrow, \neg, =, \forall, \exists, (,)$, the variables $v_0, v_1..., v_a...v_z$, the binary relationship $\in$ and the unary relationship symbols M ('is a set') and Cls ('is a class').
The reader will be familiar with the usual concepts of set theory and we will mostly follow [Gödel] with definitions and symbols. So for functions we have the range first, the domain second in the ordered pairs (x,y). $\mathcal{P}(x)$ is the powerset(-class) of x. If F is a function we write Fnc(F) with the domain Dom(F) and the range Rng(F) and if $A \subseteq Dom(F)$ we write $Rng_F(A)$ to designate the range of F confined to A. On is the class of ordinals.
As usual we use uppercase characters to identify classes and lowercase characters for symbols, which satisfy $M(x)$ (x is a set). If M is a model of NBG

set theory we will write $\in^M, V^M, L^M$ for the element relationship, class of all sets and the constructible sets. We call a formular unary, if it has exactly one free parameter. Greek letters designate logical expressions. $\tilde{\varphi}$ shall denote the formular, which is obtained from φ by restricting all quantifiers and parameters to sets and ˜shall allways indicate a formular having this property or changed to have that property.

Definition 1. *Let M be a model of NBG. A class (set) A of M is called 0-definable, if there is a unary formula φ such that $x \in A \leftrightarrow \tilde{\varphi}(x)$.*

Lemma 1. *0-definability*
If $\varphi(v_0, v_1, .., v_n)$ is a formula of $\mathcal{L}_{NBG}$ then there is for each assignment $v_1, ..v_n$ with 0-definable sets $[\psi_1], ..., [\psi_n]$ a unary formular ψ such that $\psi(x) = \tilde{\varphi}(x, [\psi_1], ..., [\psi_n])$

Proof. Let $\varphi(v_0, v_1, ...v_n)$ be a formula with v_i 0-definable sets and $\psi_i(x)$ their defining unary formulas. Define $\sigma_i(v_i) := \forall v_a : v_a \in v_i \leftrightarrow \tilde{\psi_i}(v_a)$.
Then $\psi(v_0) := \exists v_1, ...v_n\, (\sigma_1(v_1) \wedge ... \wedge \sigma_n(v_n) \wedge \tilde{\varphi}(v_0, v_1...v_n)\,)$ is an equivalent definition. □

This allows us to work with formulas having more than one parameters.

3 The Structure D

The underlying objects of the structure D are the unary formulars of $\mathcal{L}_{NBG}$. These formulars have a dual role. On one hand they are raw classes and as such their formular is just a lable for the class. We will write $[\varphi], X_\varphi \; or \; x_\varphi$ to identify these classes. On the other hand the formular provides the mapping logic to identify the elements of the class and in this role we will write $\tilde{\varphi}$, since only the bounded formular is relevant for this purpose, as we are looking for NBG.
It is important to emphasize, that in this model equal classes may not be identical: $\neg(A = B \rightarrow A \equiv B)$.
The following symbols are frequently used from now on:

$\tilde{\varphi}_0$	:=	$\neg\, x = x$
$\emptyset$	:=	$[\varphi_0]$ (empty set)
$\tilde{\varphi}_\omega$	:=	$\forall v : \big(\emptyset \in v \wedge \forall u : \big((u \in v) \rightarrow (u \cup \{u\}) \in v\big)\,\big) \rightarrow x \in v$
ω	:=	$[\varphi_\omega]$ (the integers)

The mapping rules need to define the relations $\in$ and M of $\mathcal{L}_{NBG}$. Since we want the model only to demonstrate how countably many objects can span an uncountable universe, the mapping rules will include the core axioms of set theory.

The element relationship is defined by

Rule (R0). $\forall x : x \in [\varphi] \leftrightarrow \tilde{\varphi}(x)$ *(definability)*
By this rule $[\varphi_0] = [\neg\, x = x]$ *is the empty set.*

Rule (R8). $(\neg A = \emptyset) \rightarrow \exists u : [u \in A \wedge (u \cap A) = \emptyset)]$ *(regularity)*

The unary M relationship is defined by:

Rule (R1). $(\forall x : x \in A \leftrightarrow x \in B) \leftrightarrow A = B$ *(extension)*

Rule (R2). $X \in Y \rightarrow M(X)$ *(elements are sets)*

Rule (R3). $\forall v_1, v_2 : M([v_0 = v_1 \vee v_0 = v_2]$ *(pair)*

Note that $\{v_1, v_2\}$ can be $\{v_1, v_2\}$, $\{v_1\}$, $\{v_2\}$ *or* $\emptyset$ depending on whether none, only one or both parameters are sets. This follows the practice in [Gödel].

Rule (R4). $(\neg\, \omega = \emptyset) \wedge\ M(\omega)$ *(infinity)*

Rule (R5). $\forall v_1 : M([\exists v_a : (v_0 \in v_a) \wedge (v_a \in v_1)])$ *(union set)*

Rule (R6). $\forall v_1 : M([v_0 \subseteq v_1])$ *(power set)*

Rule (R7). $\Big(Fnc(A) \wedge (\, x \subseteq\ Dom(A)\,) \Big) \rightarrow\ M(Rng_A(x)\,)$ *(substitution)*

4 D is a model of NBG less AC

Theorem 1. *D is a model of NBG less the global axiom of choice.*

The axioms of NBG are [Gödel]:

Axiom (A1). *Cls(x) (every x is a class). There is nothing to proof.*

Axiom (A2). $X \in Y \rightarrow M(X)$*. This is rule R2 of D*

Axiom (A3). $\forall u : [u \in X \leftrightarrow u \in Y] \rightarrow X = Y$ *(extension). See rule R0.*

Axiom (A4). $\forall x, y : \exists z : \forall u : [u \in z \leftrightarrow u = x \vee u = y]$ *(pairing). See rule R3 of D*

Axiom (B). *For every formular* $\tilde{\varphi}$ *which is bound to sets*[1]*:*
$\forall v_1, .., v_n : \exists A : \forall x : x \in A \leftrightarrow \tilde{\varphi}(x, v_1, ..., v_n)$*(comprehension)*

[1] instead of the axioms B1-B8 we utilize here the equivalent axiom schema from Gödel's M4

Proof. Let $v_i = [\psi_i]$. Then by lemma 1 and because of the construction of D there is a formular $\tilde{\psi}(x) = \tilde{\varphi}(x, [\psi_1], ..., [\psi_n])$. Then $A = [\psi]$ is the required class. $\square$

Axiom (C1)**.** $\exists a : \{\neg\big(a = \emptyset \wedge \forall x : [\, x \in a \to \exists y : [y \in a \wedge x \subset y]\,]\}$ *(infinity). This follows from rule R4 of D with ω being the requested set.*

Axiom (C2)**.** $\forall x : \exists y : \forall u, v : [\, u \in v \wedge v \in x \to u \in y\,]$ *(union). See rule R5*

Axiom (C3)**.** $\forall x : \exists y : [\, u \subseteq x \to u \in y\,]$ *(power set). This follows from R6.*

Axiom (C4)**.** $\forall x : \forall A : \{Fnc(A) \to \exists y : \forall u : [\, u \in y \leftrightarrow \exists v : [v \in x \wedge (u, v) \in A\,]\,]\}$ *(substitution)*

Proof. $y = Rng_A(x \cap Dom(A)) = [\exists v_1 : v_1 \in x \wedge (v_0, v_1) \in A]$ is by rule R7 a set and satisfies C4. $\square$

Axiom (D)**.** $(\neg A = \emptyset) \to \exists u\,[u \in A \wedge (u \cap A) = \emptyset)]$ *(regularity). This is rule R8 of D.*

5 Sources

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