A countable Example of Set Theory

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Introduction

Target

How countably many objects can span an uncountable universe

Mathematical base and notations

Propositional logic, $\mathcal{L}1_{\in,M}$ + Peano + NBG axioms
$\tilde{\varphi} := \varphi$ relativized to sets
Unary formula := Formula with 1 free parameter
0-definable class:=$[\varphi] = \{x|\tilde{\varphi}(x), \varphi$ unary$\}$

Examples

empty set:= $[x \neq x]$
$\{0\}$:=$[\exists v_1 : (\forall v_2 : v_2 \in v_1 \leftrightarrow v_2 \neq v_2) \wedge v_0 = v_1]$

Lemma on 0-definable Classes/Sets

Formulas with only 0-definable parms have an equivalent unary formula

For $\tilde{\varphi}$ there is $\tilde{\psi}$ such that $\tilde{\varphi}(v_0, [\psi_1], ...[\psi_n]) = \tilde{\psi}(v_0)$
with $\psi(v_0) = \exists v_1 ... v_n : \sigma_1(v_1) \wedge ... \wedge \sigma_n(v_n) \wedge \varphi(v_0, v_1 ... v_n)$
with $\sigma_i = \forall v_a : v_a \in v_i \leftrightarrow \tilde{\psi}_i(v_a)$

More examples

Let $x = [\varphi]$, $y = [\psi]$ then:
Pair: $z = [v_0 = x \vee v_0 = y]$
Union: $\bigcup x = [\exists y : (v_0 \in y) \wedge \varphi(y)])$
Power: $\mathcal{P}(x) = [v_0 \subseteq x]$
$\omega = [\forall w : (\emptyset \in w \wedge \forall u : ((u \in w) \rightarrow (u \cup \{u\}) \in w)) \rightarrow v_0 \in w]$

The model D of unary Formulas

Target

Create a countable model D for NBG -AC without ground model by using unary formulas as domain.

Domain, equality and $\in$ relationship of D

Domain consists of the unary formulars. Classes are named $[\varphi]$
Identity is not Equality: $[\varphi] = [\psi]$ iff $\forall x : \tilde{\varphi}(x) = \tilde{\psi}(x)$
Formulas have a dual role: Be a label for a class and define its elements: $x \in [\varphi] \leftrightarrow \tilde{\varphi}(x)$
Since we do not want to proof consistency: Ext, Found and the set property of Pair, Union, Power, ω, Subst (but not their existence) are part of the mapping rules.

D is a countable model of NBG

Set axioms Ext, Pair, Union, Power, Inf, Comp, Subst and Found

The requested classes are sets, if they exist in D. We only need to find formulars for them. We did for $\{x, y\}, \bigcup x, \mathcal{P}(x), \omega$ before.

Comprehension: For all $\tilde{\varphi}$ bound to sets :
$\forall v_1, .., v_n : \exists A : \forall x : x \in A \leftrightarrow \tilde{\varphi}(x, v_1, ..., v_n)$

Proof: Let $v_i = [\psi_i]$. Then $[\tilde{\psi}(x) = \tilde{\varphi}(x, [\psi_1], ..., [\psi_n])]$ is the required class.

Substitution: $\forall x : \forall A : \{Fnc(A) \rightarrow \exists y : \forall u : [\, u \in y \leftrightarrow \exists v : [v \in x \wedge (u, v) \in A]\,]\}$

Proof: $y = [\exists v_1 : v_1 \in x \wedge (v_0, v_1) \in A]$ is the desired class.